Generic features of fluctuations in critical systems

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(Received 18 September 2002; published 14 February 2003)

The probability distribution function of magnetization of critical magnetic systems is investigated with Monte Carlo simulations. Its generic features beyond the standard universality are revealed. A mean field ansatz explains the phenomena partly.

DOI: 10.1103/PhysRevE.67.026114 PACS number(s): 05.65.+b, 05.40.−a, 05.50.+q, 75.30.−m

Recently, much attention has been drawn to fluctuations of correlated systems [1–7]. In an experiment of a closed turbulent flow, it is discovered that the probability distribution function (PDF) of the power fluctuations exhibits the same form as that of the magnetization of the 2D (two-dimensional) XY model in the spin-wave regime [2]. The finite Reynolds number Re of the turbulent flow is compared with the finite size L of the XY model. The suitably rescaled PDF's with different Re's collapse onto a single curve as the data do for different L's in the XY model. The PDF's for both systems overlap at least in four orders of magnitude. The curve is non-Gaussian and with an exponential-like tail. These observations indicate that PDF's of correlated systems in different universality classes may share approximately the same form.

Based on the hint of the turbulent-flow experiment, numerical simulations for a variety of highly correlated systems, including the Ising model and some self-organized systems, have been performed and PDF's of the corresponding global observables are measured [5]. It is suggested that the PDF's of these systems indeed exhibit approximately the same form, and only minor perturbations in PDF's lead to different universality classes. Some features of this phenomenon seems even to go beyond the critical system, e.g., to the 1D and 3D XY models [8].

In Ref. [9], it is shown that for a critical system, the standard scaling form at the critical point is a sufficient condition for the data collapse of different L's. In critical phenomena, it is generally believed that PDF's are classified by universality classes [10–12]. Therefore, the suggestion in Ref. [5] seems contradicting the standard idea of universality. To examine this, simulations have been performed for the Ising and 2D XY models [7]. It is found that PDF's of these systems *at* the critical temperatures differ significantly. However, for the 2D Ising model at a certain temperature T *below* the critical temperature T_c, the PDF looks approximately the same as that of the XY model in the spin-wave regime. The PDF of the 2D Ising model at this temperature is very different from that at T_c.

Although a "superuniversal" behavior does not exist for the PDF's at T_c, it is still puzzling and very interesting as to what is hidden behind the fact that the PDF's of different systems at certain temperatures in *critical* regime join together. In this paper, critical systems with a second-order phase transition are mainly concerned, but general implication of the conclusions will be discussed. All data are obtained with standard local Monte Carlo algorithms.

We first examine and formulate more clearly and precisely the observations in Refs. [5,7]. Introducing the reduced magnetization $m=(M-\langle M\rangle)/\sigma$ with M being the magnetization, $\langle M\rangle$ being its mean and σ being the standard deviation, we denote the normalized PDF as $P(m)$. In Refs. [2,5,8], $P(m)$ of the 2D XY model in the spin-wave regime is suggested to be T and L independent. For the systems with a second-order phase transition, the PDF in critical regime depends on $K\sim 1/T$ and L through a scaling variable $s=L^{1/\nu}(K-K_c)/K_c$, and sometimes is written as $P(m,s)$ [7]. s^{ν} represents the ratio of the lattice size and the correlation length at K.

Rigorously speaking, $P(m)$ of the 2D XY model depends slightly on T in the regime close to the transition temperature T_{KT}, and it becomes T independent only in lower temperatures. In Fig. 1, stars represent $P(m)$ of the 2D XY model at $T=0.89$, just below T_{KT} [13], while squares are for $T=0.70$. As is observed in Ref. [5], the left tail of $P(m,s)$ of the 2D Ising model at $s=7.26$ (dashed line) fits nicely to the squares, but there is a deviation for the right tail. We empha-

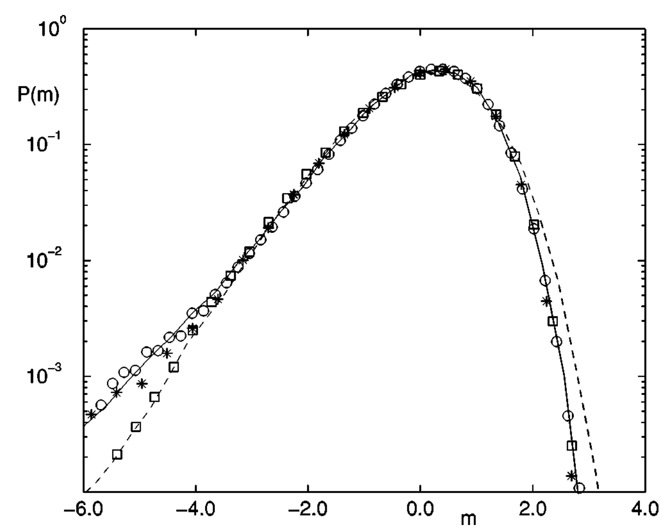

FIG. 1. Stars and squares are for the XY model with $L=64$ at $T=0.89$ and 0.70, respectively. The solid line is for the 2D Ising model with $K=0.4707$ and $L=64$, i.e., $s=4.36$ ($K_c=0.4407$), while the dashed line is with $K=0.4657$ and $L=128$, i.e., $s=7.26$. Circles are for the 3D Ising model with $K=0.2237$ and $L=32$, i.e., $s=2.21$ ($K_c=0.2217$, $\nu=0.63$ [16]).

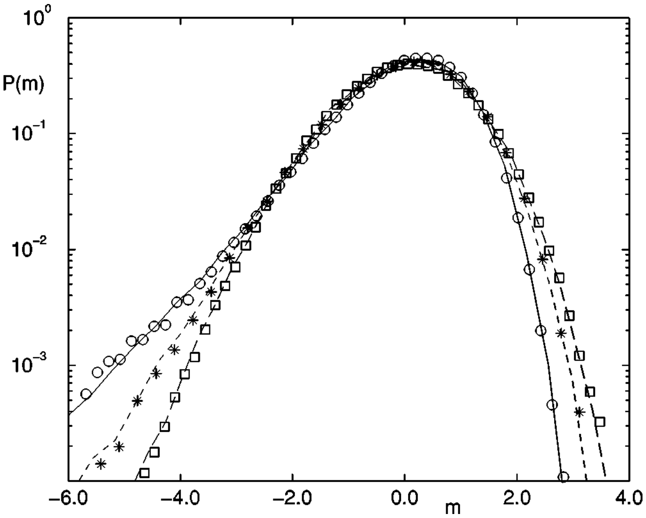

FIG. 2. The solid, dashed, and long dashed lines are for the 2D Ising model with $(K,L)=(0.4707,64)$, $(0.4707,128)$, and $(0.5157,128)$ (i.e., $s=4.36$, 8.72, and 21.58), respectively. Circles, stars, and squares are for the 3D Ising model with $(K,L)=(0.2237,32)$, $(0.2257,32)$, and $(0.2317,32)$ (i.e., $s=2.21$, 4.42, and 11.05).

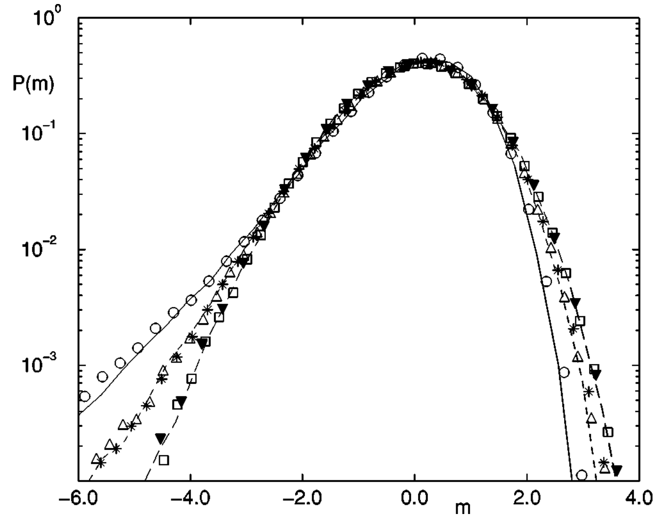

FIG. 3. The solid, dashed, and long dashed lines are the same as in Fig. 2. Circles, stars, and squares are for the 2D ϕ^4 theory with $(K,L)=(0.599,64)$, $(0.625,64)$, and $(0.625,150)$, i.e., $s=3.14$, 6.05, and 14.19 ($m_0^2=2$, $K_c=0.571$). Triangles up and filled triangles down are for the potential V with $(m_e^2,\lambda_e)=(2,0.11)$ and $(2,0.05)$.

size that the deviation is not from statistical errors or/and finite size effects. On the other hand, as is pointed out in Ref. [7], $P(m,s)$ of the 2D Ising model at $s=4.36$ (solid line) coincides better with the stars, though not perfectly for the left tail. (Data here are recorded in smaller ΔM and also more accurate than those in Ref. [7] with $s=2.90$.) More interestingly, $P(m,s)$ of the 3D Ising model at $s=2.21$ (circles) falls onto the solid line almost completely.

Is it accidental that the PDF's of the three models join at a specific curve? Our first observation is that it is *not* accidental at least for the Ising models. Actually, $P(m,s)$'s of the 2D and 3D Ising models overlap in whole large s regime. This is demonstrated in Fig. 2. $P(m,s)$'s of the 2D Ising model with $s=4.36$, 8.72, and 21.58 fit very well to those of the 3D Ising model with $s=2.21$, 4.42, and 11.05, respectively. In other words, $P(m,s)$'s of the 2D and 3D Ising models are the same in large s regime if s of the 3D Ising model is rescaled by a factor of about 2. This is a generic feature *beyond* the standard universality. The left tail of $P(m,s)$ of the 2D Ising model around $s=8.72$ looks exponential-like. $P(m,s)$ crosses over gradually to Gaussian as s increases. In the remaining part of the paper, $P(m,s)$'s of other systems will be always compared with the data of the 2D Ising model with $s=4.36$, 8.72, and 21.58.

In order to understand the generic feature of $P(m,s)$ shown in Fig. 2, we turn to the scalar ϕ^4 theory, which is believed to be the mesoscopic theory of the Ising model. The D-dimensional ϕ^4 theory falls into the universality class of the D-dimensional Ising model. It has been demonstrated in Ref. [11] that $P(m,s)$'s of both systems present the same form in small s regime. On a square or cubic lattice, the Hamiltonian of the D-dimensional ϕ^4 theory is

$$H=\sum_i\left[\frac{1}{2}\sum_\mu(\phi_{i+\mu}-\phi_i)^2-\frac{1}{2}m_0^2\phi_i^2+\frac{1}{4}\lambda_0\phi_i^4\right]. \quad (1)$$

Here μ represents unit vectors in spatial directions. The temperature $T\sim 1/K$ can be absorbed into the coupling λ_0. Therefore, $\lambda_0\sim 1/K$. Monte Carlo simulations of the ϕ^4 theory are much more time consuming than those of the Ising model. In Fig. 3, $P(m,s)$'s of the 2D ϕ^4 theory with $s=3.14$, 6.05, and 14.19 are compared with those of the 2D Ising model with $s=4.36$, 8.72, and 21.58. In Ref. [11], it is found that even in small s regime, there are corrections to scaling for $P(m,s)$ of the 2D ϕ^4 theory. Therefore, the fitting of the data for the two models in Fig. 3 is less perfect than in Fig. 2. If we rescale s of the 2D ϕ^4 theory by a factor of 1.44, the values of s in Fig. 3 become $s=4.53$, 8.72, and 20.45 and differ from those of the 2D Ising model by a few percent. Part of the deviation is also from finite size effects. Taking into account the difficulties in simulating the ϕ^4 theory, we are satisfied with the results and consider that $P(m,s)$'s of the 2D Ising model and ϕ^4 theory are the same in large s regime. This is more or less expected, since the same $P(m,s)$'s of two critical systems in small s regime should remain the same in large s regime, if effects of boundary conditions can be neglected. This result also confirms that $P(m,s)$ is indeed universal in the whole s regime for critical systems in a same universality class.

In the following, we demonstrate that a mean field ansatz

$$\phi_i\phi_{i+\mu}\sim c\,\phi_i^2, \quad c>0 \quad (2)$$

can be a good approximation at the zero order in large s regime. With this ansatz, the Hamiltonian of the ϕ^4 theory is simply an effective ϕ^4 potential

$$V=-\tfrac{1}{2}m_e^2\phi^2+\tfrac{1}{4}\lambda_e\phi^4. \quad (3)$$

We simulate this simple system also with Monte Carlo methods. As shown in Fig. 3, $P(m)$'s of V with (m_e^2,λ_e)

$=(2,0.11)$ and $(2,0.05)$ fit well to those of the 2D Ising model with $s=8.72$ and 21.58, respectively. No values of m_e^2 and λ_e can produce $P(m,s)$ of the 2D Ising model at $s=4.36$. However, it is interesting that $P(m)$ with $(m_e^2,\lambda_e)=(2,1)$ coincides with that of the 3D Ising model at T_c. If we consider $(m_e^2,\lambda_e)=(2,1)$ as a "critical" point, $(m_e^2,\lambda_e)=(2,0.11)$ and $(2,0.05)$ correspond to $s=8$ and 20. Since $21.58/8.72\simeq 20/8\sim 2.5$, $P(m,s)$ of V is the same as that of the 2D Ising model in sufficiently large s regime, e.g., $s\geqslant 8.72$.

In sufficiently large s regime, the correlation length is much smaller than the lattice size. Fluctuations are less important and the effective potential dominates the behavior of $P(m,s)$. For example, in the simulations of the ϕ^4 theory at $s=8.72$ and 21.58, we observed that with $L=32$ or 64, only a few percent of the variables ϕ_i's take negative values when the magnetization M drops into the positive sector of the phase space. Most ϕ_i's fluctuate around its mean value in the positive sector. The mean field ansatz makes sense. For larger lattices, more ϕ_i's become negative, but the standard scaling form keeps $P(m,s)$ with a fixed s unchanged. Around $s\sim 4.36$, the system starts the crossover to large fluctuation regime (comparing the correlation length and the lattice size) and the mean field ansatz fails.

It is shown in Ref. [5] that $P(m)$ of the 2D XY model presents exponential behavior for the left tail and double exponential for the right tail. If we define $\varphi=\phi-(m_e^2/\lambda_e)^{1/2}$, $V=m_e^2\varphi^2+(m_e^2\lambda_e)^{1/2}\varphi^3+1/4\lambda_e\varphi^4$. Since the mean value $\langle\phi\rangle$ is close to $(m_e^2/\lambda_e)^{1/2}$, the "reduced magnetization" $m\sim\varphi/\sigma$ with σ being the standard error. Due to the cubic term of φ, the left tail of $P(m)$ of V around $(m_e^2,\lambda_e)=(2,0.11)$ looks approximately exponential-like up to $m\sim 8$. The right tail of $P(m)$ of V decays slower than double exponential and faster than Gaussian, and this is in agreement with the curve of the Ising model in Fig. 3, and as shown also in Fig. 2 in Ref. [5].

To show how generic the form of $P(m,s)$ of the Ising model in large s regime is, we have finally performed simulations for the 2D three-state Potts model. In Fig. 4, $P(m,s)$'s with $s=4.21$, 10.95, and 27.70 are compared with those of the 2D Ising model with $s=4.36$, 8.72, and 21.58, respectively, and they overlap each other. Since $27.70/10.95\simeq 21.58/8.72\sim 2.5$, we conclude that up to a constant factor of s, $P(m,s)$ of the Potts model with $s\geqslant 10.95$ is the same as that of the Ising model with $s\geqslant 8.72$. $P(m,s)$ of the Potts model in the regime around $s\sim 4.21$ is the same as that of the Ising model around $s\sim 4.36$ only up to a nonlinear transformation of s.

This is an interesting result. In a small s regime, $P(M,s)$'s of the Ising and Potts models are very different. [Note that $P(M,s)$ is not the same as $P(m,s)$.] Actually, $P(M,s)$ of the Potts model is even not symmetric in M. In a large s regime, however, $P(m,s)$'s of both systems tend to be identical. This indicates that in *critical* regime, the effective potential V in Eq. (3) is rather generic, and other possible terms are irrelevant in the sense of the renormalization group transformation. In Fig. 2 in Ref. [5], $P(m)$'s of at least, the 2D site percolation model, the granular media model and the correlated extremal process all show a slower right tail than double exponential. They should fit better to $P(m)$ of V at suitable values of m_e^2 and λ_e. Due to fluctuations, it is not clear whether the right tail of the turbulent flow is double exponential or slower [2].

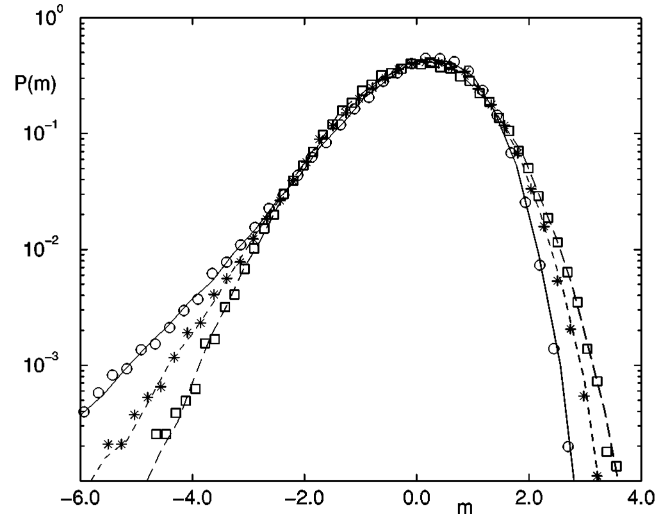

FIG. 4. The solid, dashed, and long dashed lines are the same as in Fig. 2. Circles, stars, and squares are for the 2D Potts model with $(K,L)=(1.030,72)$, $(1.070,72)$, and $(1.070,156)$ (i.e., $s=4.21$, 10.95, and 27.70).

The 2D $O(2)$ model falls into the same universality class of the 2D XY model. The potential of the $O(2)$ model is also ϕ^4 like. Due to the rotational degree of freedom, however, the system undergoes a Kosterlitz-Thouless phase transition. Below the transition temperature T_{KT}, it remains critical. Therefore, $P(m)$ of the XY model at low temperatures is not fully governed by V, and shows only some similar features as shown in Fig. 1.

In conclusion, for critical systems with a second-order phase transition, the probability distribution function $P(m,s)$ exhibits a rather generic form in large s regime. In sufficiently large s regime, e.g., $s\geqslant 8.72$ for the 2D Ising model, $P(m,s)$'s of all the models examined in this paper share the same form (up to a constant factor of s). This form of $P(m,s)$'s can be described by an effective potential V in Eq. (3), which corresponds a mean field Ansatz of the ϕ^4 theory. $P(m)$'s of some other correlated systems discussed in Ref. [5] should also fall into this form. $P(m)$ of V around $(m_e^2,\lambda_e)=(2,0.11)$ shows approximately exponential-like behavior for the left tail. This characterizes the PDF's of the 2D XY model in the spin-wave regime and the closed turbulent flow.

In medium-large s regime, e.g., $s\sim 4.36$ for the 2D Ising model, $P(m,s)$'s of different systems split according to symmetries. But $P(m,s)$ of the 2D Potts model can still be projected approximately onto that of the Ising model through a nonlinear transformation of s. In this regime, the systems start the crossover to large fluctuation regimes and the mean field ansatz fails. However, it is interesting why $P(m,s)$ remains relatively generic, e.g., dimension independent. More understanding is needed here.

For smaller s, $P(m,s)$ will become dimension dependent and will be classified finally by standard universality classes. What is the deeper physical impact of the generic feature of $P(m,s)$ in large s regime, and whether and how it classifies critical systems beyond standard universality classes, are very important open problems. Study of the PDF $P(m,s,t)$ in nonequilibrium dynamics [14,15] is undergoing.

Work was supported in part by NNSF (China) under Grant No. 10275054, the specialized research fund for the doctoral program of high education provided by Education Ministry (China), and DFG (Germany) under Project No. TR 300/3-3.

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